

S_n : the symm gp.

Recall! $S_n = \{p, \text{permutations of } \{1, 2, \dots, n\}\}$
 $|S_n| = n!$

Given $p \in S_n$ a permutation $\leadsto L(p): \mathbb{R}^n \rightarrow \mathbb{R}^n$.

Let $e_1, e_2, \dots, e_n$ be the std basis of $\mathbb{R}^n$

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \end{pmatrix}, \dots, e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \end{pmatrix}.$$

Define $L(p)$ as

$$L(p)(e_j) = e_{p(j)} \quad \text{for } j = 1, 2, \dots, n.$$

Then extend $L(p)$ to all of $\mathbb{R}^n$ by linearity.

$$\text{I.e. } L(p)(c_1 e_1 + c_2 e_2 + c_3 e_3) \quad c_1, c_2, c_3 \in \mathbb{R}$$

$$= c_1 L(p)(e_1) + c_2 L(p)(e_2) + c_3 L(p)(e_3)$$

$$= c_1 e_{p(1)} + c_2 e_{p(2)} + c_3 e_{p(3)}$$

$$p: \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n\}$$

Ex $S_3 \ni \chi = (123)$

$$L(p) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

if $y = (12)$ then $L(y) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Propⁿ Suppose $p_1, p_2 \in S_n$. Then

$$L(p_1 p_2) = L(p_1) L(p_2) \quad \forall p_1, p_2 \in S_n.$$

Pf: Suffices to check agreement on $e_j, 1 \leq j \leq n$

$$\bullet L(p_1 p_2)(e_j) = e_{p_1 p_2(j)} = e_{p_1(p_2(j))}$$

$$\bullet L(p_1) L(p_2)(e_j) = L(p_1)(e_{p_2(j)})$$

$$= e_{p_1(p_2(j))}$$

✓
□

Defⁿ of sign of a permutation

Lemma: $p \in S_n$. Then $\det(L(p)) = \pm 1$.

Pf: HW question.

Defⁿ $\text{sgn}: S_n \rightarrow \{\pm 1\}$ is defined as

$$\text{sgn}(p) = \det(L(p))$$

Claim: $\text{sgn}(p_1 p_2) = \text{sgn}(p_1) \text{sgn}(p_2)$

$p_1, p_2 \in S_n$
comp.
of
permutations

product of elts of $\{\pm 1\}$

$1, 2, \dots, n$
of
permutations

product of eels of $\{\pm 1\}$

Pf:
$$\begin{aligned} \text{sgn}(p_1 p_2) &= \det(L(p_1 p_2)) = \det(L(p_1) L(p_2)) \\ &= \det(L(p_1)) \det(L(p_2)) \\ &= \text{sgn}(p_1) \text{sgn}(p_2) \end{aligned}$$

Defⁿ $A_n \subseteq S_n$ define to be the $p \in S_n$ with $\text{sgn}(p) = 1$.

A_n is called the alternating gp of n eels
or the gp of even permutations of $\{1, \dots, n\}$.

Example $x \in S_3$ (123)

$$L(x) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\det(L(x)) = 1 \Rightarrow \text{sgn}(x) = 1$$

$$L(y) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\det(L(y)) = -1 \Rightarrow \text{sgn}(y) = -1.$$

$$A_3 \subseteq S_3 \text{ is } A_3 = \{1, x, x^2\}$$

$1, x, x^2$ have $\text{sgn} = 1$

and y, xy, x^2y have $\text{sgn} = -1$.

Claim: $A_n \subseteq S_n$ is a subgp.

- $p_1, p_2 \in A_n \Rightarrow \text{sgn}(p_1 p_2) = \text{sgn}(p_1) \text{sgn}(p_2)$

$$\bullet \quad p_1, p_2 \in A_n. \Rightarrow \text{sgn}(p_1 p_2) = \text{sgn}(p_1) \text{sgn}(p_2) \\ = 1 \cdot 1 = 1$$

$$\bullet \quad 1 \in A_n: \quad L(1) = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \Rightarrow \text{sgn}(1) = 1.$$

$$\bullet \quad \text{If } p \in A_n \text{ then } 1 = \text{sgn}(1) = \text{sgn}(p p^{-1}) = \text{sgn}(p) \text{sgn}(p^{-1}) \\ = 1 \cdot \text{sgn}(p^{-1}) = \text{sgn}(p^{-1})$$

$$\Rightarrow p^{-1} \in A_n. \quad \#$$

Klein four gp abⁿ gp of size 4

$$\bullet \quad K_4 = \left\{ \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \right\}$$

$$\text{If } x \in K_4 \text{ then } x^2 = 1.$$

$$\bullet \quad SU(2) = \left\{ \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix} : \alpha, \beta \in \mathbb{C} \text{ s.t. } |\alpha|^2 + |\beta|^2 = 1 \right\}.$$

$*$: complex conjugate

Exercise: $SU(2)$ is a gp, a subgp of $GL_2(\mathbb{C})$

$H \subseteq SU(2)$ the quaternion gp
non-abelian gp of order 8

$$H = \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \pm \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \pm \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \right\}$$

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$$

H is a subgroup of $SU(2)$.

Cyclic Gps

G a gp, $x \in G$ eff.

$$H = \{x^n : n \in \mathbb{Z}\} \subseteq G \text{ is a subgroup.}$$

It is

- smallest subgroup of G containing x
- called the cyclic subgroup of G containing x
- denoted $\langle x \rangle$
- sometimes fte, sometimes infinite

Propⁿ $\langle x \rangle \subseteq G$ the cyclic subgroup of G gen'd by x ,

$$S = \{r \in \mathbb{Z} : x^r = 1\}.$$

✓ 1) $S \subseteq (\mathbb{Z}, +)$ is a subgroup

✓ 2) $x^r = x^s \iff r - s \in S$.

3) Suppose $S \neq \{0\}$ then $S \cong \mathbb{Z}_n$ for some $n > 0$.

Then $\langle x \rangle = \{1, x, x^2, \dots, x^{n-1}\}$ has order n .

Pf: 1) If $a, b \in S$ then $x^{a+b} = x^a x^b = 1 \cdot 1 = 1$

$\implies a+b \in S$. Also S is closed under inverses, and contains 0

$\implies S \subseteq (\mathbb{Z}, +)$.

and contains $\cup$

$$\Rightarrow S \subseteq (\mathbb{Z}, +).$$

$$2) \quad x^r = x^s \Leftrightarrow \underset{\substack{\parallel \\ x^{r-s}}}{x^r x^{-s}} = x^s x^{-s} = 1 \Leftrightarrow r-s \in S.$$

3) $\left\{ \begin{array}{l} \bullet \text{ the elts } 1, x, x^2, \dots, x^{n-1} \text{ are distinct} \\ \bullet \text{ every power of } x, x^k, \text{ is one of } 1, x, x^2, \dots, x^{n-1} \end{array} \right.$

need to check

$\bullet$ Suppose $x^i = x^j$ for $i, j \in \{0, 1, 2, \dots, n-1\}$

then $i-j \in S \cong \mathbb{Z}_n$

which can only happen if $i=j$

$\bullet$ With $k = qn + r$ with $q \in \mathbb{Z}$, $0 \leq r \leq n-1$.

$$x^k = x^{qn+r} = x^{qn} x^r = (x^n)^q x^r$$

$$= 1 \cdot x^r = x^r.$$

$$x^{qn} = 1 \text{ because } qn \in \mathbb{Z}_n = S$$

$\square$

Homomorphisms: How to compare different grps

G, G' grps

$\varphi: G \rightarrow G'$ is said to be a homom.

if $\underbrace{\varphi(a \cdot b)}_{\text{product in } G} = \underbrace{\varphi(a) \varphi(b)}_{\text{product in } G'} \quad \forall a, b \in G$

Example

$$\Rightarrow \varphi(1_a) = 1_{C'} \quad \checkmark.$$

$$3) \quad 1_{C'} = \varphi(1_C) = \varphi(a a^{-1}) = \varphi(a) \varphi(a^{-1})$$

$$\Rightarrow \varphi(a^{-1}) = \varphi(a)^{-1}. \quad \square$$